

6. RULER AND COMPASS CONSTRUCTABILITY

§6.1. Ruler and Compass Constructions

The Greeks considered arithmetic and geometry as being two ways of looking at the same number system and so geometrical constructions were considered very natural to perform arithmetic operations. Since the only numbers they could conceive of arithmetically were rational numbers they assumed that that is all that could be obtained geometrically.



So it came as a great shock for them to discover that $\sqrt{2}$ is irrational, that is there's a number which has a proper geometric existence but doesn't have a proper arithmetic one. But then gradually irrational numbers came to be accepted, though they didn't lose their geometric flavour. Number was always considered to be a geometric concept. And it was accepted as an unwritten axiom that only numbers which arose in the context of geometric constructions could possibly exist.

Certain problems were posed where it was felt that a geometric construction must surely exist. It was just a matter of finding it. The three most famous were the

doubling of a cube, the trisection of an angle and the squaring of a circle. The numbers involved in these three problems clearly exist but they thought there had to be a corresponding construction. And by ‘construction’ they meant a construction using ruler and compass. It’s interesting that their intuition must have been using some primitive notion of continuity – a concept that took almost another couple of thousand years to become fully developed.

There’s a legend that when asked for a way to stop a plague that was attacking the city of Delos, the Oracle of Delphi advised that the altar of Apollo should be doubled in size. The altar was in the shape of a cube and although its sides were doubled the plague continued. The Oracle then revealed that the citizens of Delos had not done as instructed since they had increased the altar eight-fold. What was required was to *double* the volume. The problem of doubling the cube is thus:

DOUBLING THE CUBE

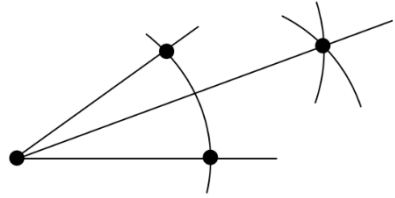
Given the side of a cube, construct the side of another cube with twice the volume.

Of course that means to construct a line that’s the cube root of 2 times the length of a given line.

TRISECTING AN ANGLE

Divide any given angle into three equal pieces.

There's a fairly simple ruler and compass construction for bisecting any given angle.



So it seems quite a natural assumption that there should be a ruler and compass construction for *trisecting* any given angle. After all, we can trisect any interval by ruler and compass (see construction (6) below).

For certain special angles, such as 90° , it can easily be done (a 30° angle can easily be constructed). But the problem is to do it for *any* given angle.

SQUARING A CIRCLE

Construct a square whose area is exactly equal to that of a given circle.

In other words, given a certain length that could be the radius of a given circle, the problem is to construct a line whose length is π times that length.

The methods allowed in all these constructions are the use of a ruler and compass. The compass is to be used to draw circles through given points and passing through others. The ruler must be used solely as a straight-edge for joining points by straight lines, not for measurement.

For this reason, ruler and compass constructions are often called constructions by *straight-edge* and compass.

The reason for disallowing measurement is the question of accuracy. The accuracy to which we can measure lengths is limited by the scale of the markings. Even if we had the means to magnify the scale we would have to end up making judgements. The whole philosophy of ruler and compass constructions is to have a procedure that is theoretically exact.

There do exist ruler and compass methods for getting quite good approximations to the solutions to all three problems. But that's not the point. With constructions representing numbers it's no good saying that these numbers *approximately* exist. We need construction methods that are *exact*. And it's been shown that all three problems are insoluble by ruler and compass methods. The geometric concept of number in terms of constructability thus proved to be inadequate and so the more general concept of real numbers gradually emerged.

Given a set of geometric objects (points, lines and circles) a ruler and compass construction is a sequence of steps each of which is one of the following.

- (1) Given two distinct points A, B, draw **LINE(A, B)**, the straight line through them.
- (2) Given two distinct points A, B draw **CIRCLE(A, B)**, the circle with centre A, passing through B.

(3) Take the point(s) of intersection of geometric objects already constructed.

For example $P = \text{LINE}(A, B) \cap \text{LINE}(C, D)$ is the point of intersection of two non-parallel lines. And $P, Q = \text{CIRCLE}(A, B) \cap \text{LINE}(C, D)$ are the points of intersection of a circle with a line. Of course this presupposes that the circles do intersect.

(4) Choose an arbitrary point on a geometric object already constructed.

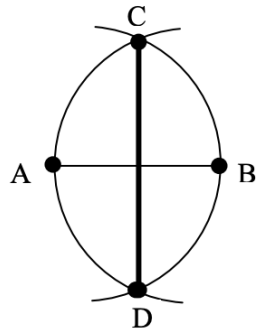
For example: **CHOOSE** $P \in \text{LINE}(A, B)$.

The ruler (straight edge) and compass are *idealised* instruments in that the points A, B in (1) and (2) can be arbitrarily far apart. An actual ruler has a finite length and an actual compass can only open so far.

§6.2. Examples of Ruler and Compass Constructions

(1) **Construct the perpendicular bisector of an interval AB .**

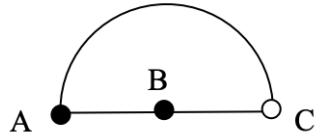
Given distinct points A, B draw $\text{CIRCLE}(A, B)$ and $\text{CIRCLE}(B, A)$. Take C, D as the two points of intersections of these circles. Then the line $\text{LINE}(C, D)$ is the perpendicular bisector of the interval AB . Note that this is also a construction for an equilateral triangle on a given base



because the triangle ABC is equilateral. The intersection of CD with AB is the midpoint of AB.

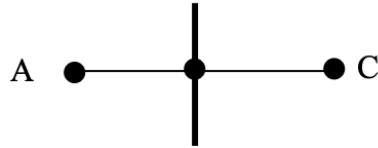
(2) Construct the reflection of a point A in a point B.

Construct the circle $CIRCLE(B, A)$. One of the points of intersection of AB with this circle is A. Let the other one be C. This is the required reflection.



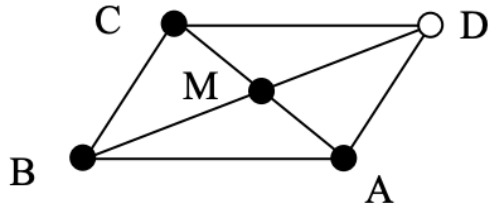
(3) Construct the perpendicular to an interval at one of the endpoints.

Construct the reflection C of A in B and construct the perpendicular bisector of AC.



(4) Given a line and a point that does not lie on that line, construct the line through the given point parallel to the given line.

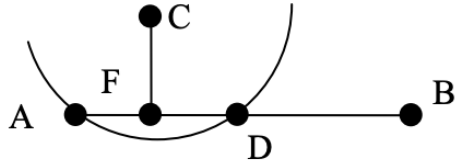
Suppose the given line is $LINE(B, A)$ and the given point is C. Construct the midpoint M of AC and reflect B in M to D. Since the diagonals of a



parallelogram bisect one another ABCD will be a parallelogram. Then $LINE(C, D)$ is the required line. Note this construction can be used to bisect angles, since the diagonals of a parallelogram bisect its four angles.

(5) Construct the foot of the perpendicular from C to AB.

Draw the circle $CIRCLE(C, A)$. The point A will be one of the points of intersection of this circle with AB. Let the other one be D. Then the midpoint F of AD will be the required foot.



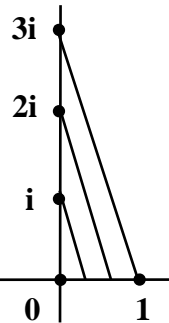
NOTE: This fails if B happens to be the foot of the perpendicular already for then there is only one point of intersection of the circle with the line.

(6) Divide any interval into any number of equal parts.

To divide the interval from 0 to 1 into n parts we construct the line l joining ni on the imaginary axis to 1 on the real axis. Then we draw lines through the points

$$i, 2i, \dots, (n - 1)i$$

on the imaginary axis, parallel to the line l . The points where these lines intersect the real axis divide the interval from 0 to 1 into n equal parts.



More on Ruler and Compass Constructions can be found in my notes on *Geometry Volume 2*.

§6.3. Constructible Numbers

We identify a complex number with its corresponding point on the complex plane. A **constructible number** is a complex number whose corresponding point on the complex plane can be constructed by ruler and compass, starting with two points representing 0 and 1. We can easily construct the real and imaginary axes and the number i . Then we can divide the interval from 0 to 1 in any number of equal parts. By reflecting points in other points we can construct all the rational numbers and all the rational multiples of i . Then by constructing suitable rectangles we can construct all complex numbers with rational real and imaginary parts.

Clearly a complex number is constructible if and only if its real and imaginary parts are constructible.

Theorem 1: The distance between two constructible points is constructible.

Proof: Suppose P , Q are constructible. Let O be the origin.

Case I OPQ are non-collinear:

Let d be the distance OR . Construct the parallelogram $OPQR$. Then the length of OR is also d . Draw the $\text{CIRCLE}(O, R)$ and intersect it with the real axis. One of the two points of intersection will represent d . (The other will represent $-d$.)

Case II: OPQ are collinear:

Choose R not on LINE(O, P). Construct the parallelogram PQRS. ORS are not collinear and so we can use case I.

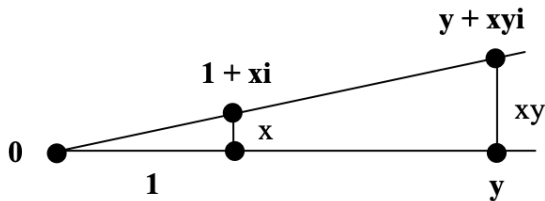
Theorem 2: The set of constructible numbers is a field

Proof: We begin by showing that the set of all positive real constructible numbers is closed under addition, multiplication and division. Let x, y be positive real numbers. We can construct $-x$ by reflecting x in the origin. The distance between $-x$ and y is $x + y$ so by Theorem 1, $x + y$ is constructible.

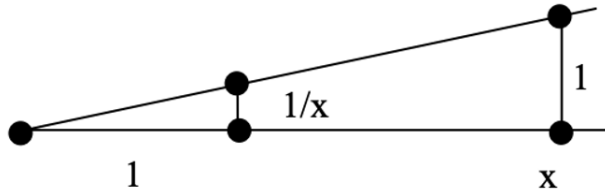
If $x > y$ the distance between them is $x - y$ and so is constructible.

If $x < y$ the distance between them is $y - x$ and, reflecting this in the origin, we'll have constructed $x - y$.

For products and quotients use similar triangles. Construct the point $1 + xi$. The line LINE(0, $1 + xi$) and the vertical at y intersect at the point $y + xyi$. The distance between y and $y + xyi$ is xy .



Similarly we can construct $1/x$ if $x > 0$.

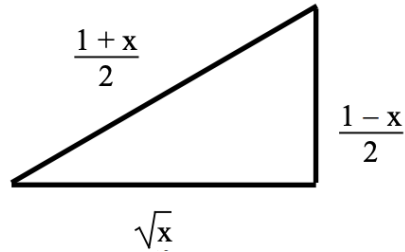


Since the sum, product and additive and multiplicative inverses of a complex number can be expressed in terms of their real and imaginary parts, using only the field operations of addition, subtraction, multiplication and division, the theorem follows from the real case and Theorem 1.

Corollary: Rational numbers are constructible.

Theorem 3: The square roots of a constructible number are constructible.

Proof: The theorem is true for positive real constructible numbers as the following construction shows (we leave it as an exercise to obtain an explicit construction).



For a non-real constructible number, $z = re^{i\theta}$ we construct $r = \text{CIRCLE}(0, z) \cap \text{LINE}(0, 1)$. Then we construct \sqrt{r} as above. To get the square roots of z we bisect the angle

between the real axis and $\text{LINE}(0, z)$ and intersect this line with the circle $\text{CIRCLE}(0, \sqrt{r})$.

Example 1:
$$\sqrt{3 + \sqrt{5 - \sqrt{\frac{3 + \sqrt{2}}{\sqrt{7}}}}} + \frac{\sqrt[4]{5}}{\sqrt{3 + \sqrt{2}}} i$$

is constructible (though it would probably be a nightmare to obtain an explicit construction).

While there are infinitely many constructible numbers, of arbitrary complexity, most complex numbers are not constructible. In the course of this chapter we'll prove that $\sqrt[3]{2}$ and $e^{2\pi i/9}$ are not constructible, thereby proving the impossibility of doubling the cube and trisecting a given angle, by ruler and compass. Another famous non-constructible number is π whose non-constructability establishes the impossibility of squaring the circle.

Theorem 4: If a point (α, β) is constructible by ruler and compass with rational coordinates, the degrees of α and β over \mathbb{Q} are powers of 2.

Proof: Suppose that at some stage in the ruler and compass construction the coordinates of all points generate some number field F . Then any point (α, β) which can be constructed from these points in one step is a point of intersection of two curves of the form:

$$ax^2 + ay^2 + 2fx + 2gy + c = 0.$$

For $a \neq 0$ this represents a circle with centre $(-f/a, -g/a)$

$$\text{and radius } \sqrt{\frac{f^2}{a^2} + \frac{g^2}{a^2} - c}.$$

For $a = 0$ it represents a straight line.

The coefficients of the equations are expressible in terms of the coordinates of the points from which the circles/lines were constructed using only the operations of addition, subtraction, multiplication and division and so they belong to F .

Eliminating y from these two equations we find that α is a zero of some quadratic with coefficients in F . Thus the degree of the minimum polynomial of α over F is 1 or 2. Hence $|F[\alpha]/F| = 1$ or 2. Similarly $|F[\beta]/F| = 1$ or 2.

As the ruler and compass construction proceeds we build up a sequence of fields, each having degree 2 over the previous one. By Theorem 7 the degree of each of these fields over \mathbb{Q} must be a power of 2.

If α is now a coordinate of any point which is constructible by ruler and compass (starting with points with rational coordinates) then $\alpha \in F$ for some number field with $|F/\mathbb{Q}| = 2^n$ for some n . By Theorem 7 $|F/\mathbb{Q}[\alpha]| \cdot |\mathbb{Q}[\alpha]:\mathbb{Q}| = |F:\mathbb{Q}| = 2^n$ and so $|\mathbb{Q}[\alpha]/\mathbb{Q}|$ is a power of 2.

Theorem 5: There is no ruler and compass construction that exactly trisects any given angle.

Proof: Suppose such a construction exists. We can certainly construct an angle of 60° by ruler and compass and so we would be able to construct a 20° angle, making $2\cos(\pi/9)$ a constructible number.

Let $c = \cos(2\pi/9)$ and $s = \sin(2\pi/9)$. Then $(c + is)^3 = c^3 + 3ic^2s - 3cs^2 - is^3$.

But by De Moivre's Theorem

$$(c + is)^3 = \cos(2\pi/3) + i \sin(2\pi/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Equating real parts we get $c^3 - 3c(1 - c^2) = -1/2$ and so $8c^3 - 6c + 1 = 0$.

Let $\alpha = 2c$.

Then α is a zero of $x^3 - 3x + 1$.

Modulo 2 this becomes $x^3 + x + 1$, which is prime over \mathbb{Z}_2 and so $x^3 - 3x + 1$ is prime over \mathbb{Q} .

Hence the minimum polynomial of $2\cos(\pi/9)$ is:

$$x^3 - 3x + 1$$

and therefore $|\mathbb{Q}[\cos(2\pi/9)]/\mathbb{Q}| = 3$. Since this is not a power of 2 we get a contradiction.

An ellipse can be constructed by string and pins, by tying the ends of the string and looping it around two pins that have been hammered into the paper on a drawing board. Of course circles can be drawn with string and pins using just one pin.

In my notes *Geometry Volume 1* I prove that any angle can be trisected using string and pins, and a straight edge. But any construction that involves numbers α where $|\mathbb{Q}[\alpha]/\mathbb{Q}|$ is not of the form $2^a 3^b$ is not possible using such methods.

EXERCISES FOR CHAPTER 6

Exercise 1: Outline a ruler and compass constructions for the inscribed circle for a triangle ABC .

Exercise 2: Find a ruler and compass construction for finding the square root of an arbitrary positive real number x .

HINT: Use the identity $x = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2$

Exercise 3: Outline a ruler and compass construction for a regular pentagon.

Exercise 4: Prove that the volume of a cube cannot be doubled by a ruler and compass construction. That is, show that $\sqrt[3]{2}$ is not constructible.

Exercise 5: Investigate the following ruler and compass construction designed to trisect any given angle.

To trisect the angle AOB, draw a circle with centre O. Slide the ruler until the length CD is equal to the radius of the circle. The angle DCO is now one third of the angle AOB.

What's wrong with it?

Exercise 6: Prove that the regular heptagon (seven equal sides) is not constructible by ruler and compass.

SOLUTIONS FOR CHAPTER 6

Exercise 1: The centre of the inscribed circle is the common intersection of the perpendicular bisectors of the sides.

Exercise 2: Construct the points P, Q corresponding to $\frac{x-1}{2}$ and $\frac{x+1}{2}$ respectively.

Construct the perpendicular PY at P.

Let $R = C(O, P) \cap OY$. Then the length of OR is \sqrt{x} .

Let $S = C(O, R) \cap OP$.

Then S represents \sqrt{x} .

Exercise 3: Let $c = \cos(2\pi/5)$ and $s = \sin(2\pi/5)$.

Then $(c + is)^5 = 1$.

Equating imaginary parts we get $5c^4s - 10c^2s^3 + s^5 = 0$.

Since $s \neq 0$, $5c^4 - 10c^2s^2 + s^4 = 0$.

Hence $5c^4 - 10c^2(1 - c^2) + (1 - c^2)^2 = 0$.

Hence $16c^4 - 12c^2 + 1 = 0$.

Let $C = 2c$. Then $C^4 - 3C^2 + 1 = 0$.

So $C^2 = \frac{3 \pm \sqrt{5}}{2}$ and hence $C = \pm \sqrt{\frac{3 \pm \sqrt{5}}{2}}$.

By considering approximations we see that in fact

$$C = \sqrt{\frac{3 - \sqrt{5}}{2}}.$$

By Exercise 13 we can construct $\sqrt{5}$ and hence $\frac{3 - \sqrt{5}}{2}$.

By Exercise 13 again we can construct $\sqrt{\frac{3 - \sqrt{5}}{2}}$.

Suppose P is the point on the real that represents

$$\sqrt{\frac{3 - \sqrt{5}}{2}} = 2\cos(2\pi/5).$$

Construct a perpendicular PY to OP at P.

Let Q be the point on the imaginary that represents $2i$.

Construct a perpendicular QX to OQ at Q.

Let $R = PY \cap QX$. Then $\angle ROP = 2\pi/5$.

Let A be the point on the real axis representing the number 1.

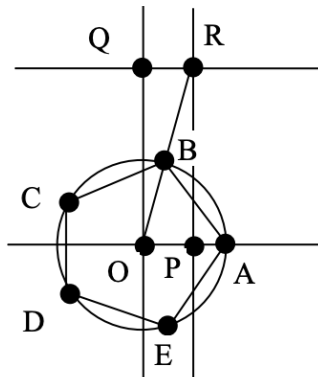
Let $B = C(O, A) \cap OR$. Then B represents $\cos(2\pi/5)$.

Let $C = C(O, A) \cap C(B, A)$ with $C \neq A$.

Let $D = C(O, A) \cap C(C, B)$ with $D \neq C$.

Let $E = C(O, A) \cap C(D, C)$ with $E \neq D$.

Then $ABCDE$ is a regular pentagon.



There is in fact a simpler construction.

Let P be the point 1 on the real axis.

Construct the perpendicular OY to OP at O .

This is the imaginary axis.

Let $B = C(O, P) \cap OY$. This is the point on the imaginary axis representing the number i .

Let $Q = C(O, P) \cap OP$ with $Q \neq A$. This represents -1 .

Let M be the midpoint of QO .

Let $R = C(M, B) \cap OP$.

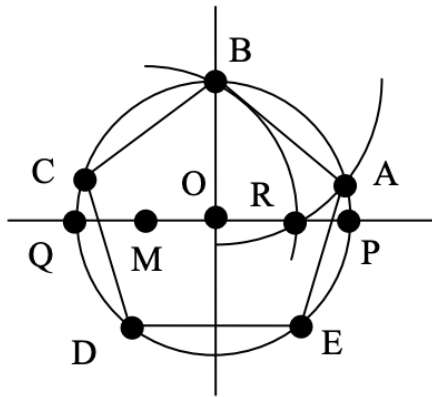
Let $A = C(O, P) \cap C(B, R)$ in the first quadrant. This represents $e^{2\pi i/5}$.

Let $C = C(O, P) \cap C(B, A)$ with $C \neq A$.

Let $D = C(O, P) \cap C(C, B)$ with $D \neq B$.

Let $E = C(O, P) \cap C(D, C)$ with $E \neq C$.

Then $ABCDE$ is a regular pentagon.



Exercise 4: The minimum polynomial of $2^{1/3}$ over \mathbb{Q} is $x^3 - 2$. Since its degree over \mathbb{Q} is not a power of 2, $\sqrt[3]{2}$ is not constructible.

Exercise 5: The procedure does, in theory, trisect $\angle AOB$ exactly, but sliding the ruler is not an allowable ruler and compass construction.

Exercise 6: Let $\alpha = e^{2\pi i/7}$.

If a regular heptagon could be constructed by ruler and compass then one could construct the angle $2\pi/7$ and so α would be a constructible number.

But α is a zero of the polynomial:

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1).$$

Since $\alpha \neq 1$, α is a zero of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

If this is the minimum polynomial we are finished, because 6 is not a power of 2.

We must show that $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ is prime over \mathbb{Q} and this could prove difficult by the usual methods since its degree is not small. For example the Too Many Primes Test would require as many as 13 prime or ± 1 values.

Eisenstein comes to our rescue. But there is no prime dividing any of the coefficients! True. But watch how we can tweak Eisenstein's Criterion.

If $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ is composite, then so is $f(x + 1)$.

$$\begin{aligned}\text{But } f(x + 1) &= (x + 1)^6 + (x + 1)^5 + (x + 1)^4 + (x + 1)^3 \\ &\quad + (x + 1)^2 + (x + 1) + 1 \\ &= x^6 + 7x^5 + 21x^4 + 35x^3 + 35x^2 + 21x + 7.\end{aligned}$$

By Eisenstein this is prime over \mathbb{Q} and hence so is $f(x)$.

